

Proving that a function is increasing/decreasing on an interval.

Consider the function f defined as $f(x) = x^2 - 6x + 2$. Completing the square yields

$$f(x) = (x - 3)^2 - 7$$

from which we see that the minimum value of the function is -7 and that it occurs when $x = 3$. That is, the point $(3, -7)$ is the lowest point on the function's graph.

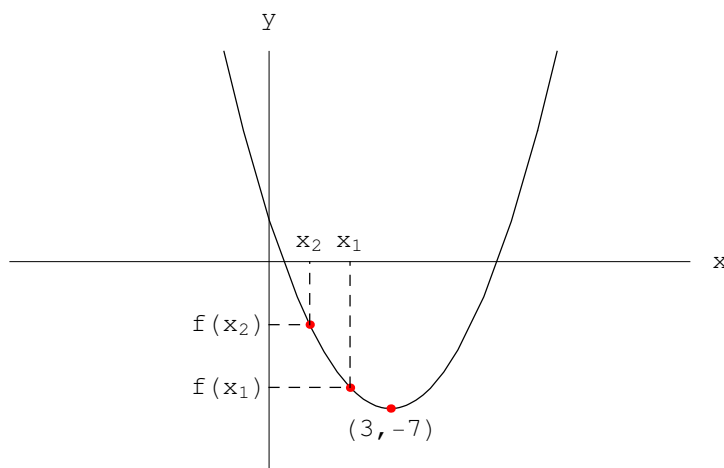
When we say that a function f is a decreasing function of x on an interval (a, b) , we mean that as x increases while remaining between a and b , $f(x)$ decreases.

When we say that a function f is an increasing function of x on an interval (a, b) , we mean that as x increases while remaining between a and b , $f(x)$ increases.

The graph of f suggests that in the interval $(-\infty, 3)$ as x increases, $f(x)$ decreases; and that in the interval $(3, \infty)$ as x increases, $f(x)$ increases. Thus, f is decreasing on $(-\infty, 3)$ and f is increasing on $(3, \infty)$.

Now, suppose we wish to prove that f defined by $f(x) = (x - 3)^2 - 7$ is decreasing on $(-\infty, 3)$. Our strategy is to show that for any x_1, x_2 where $x_2 < x_1 < 3$, $f(x_2) > f(x_1)$.

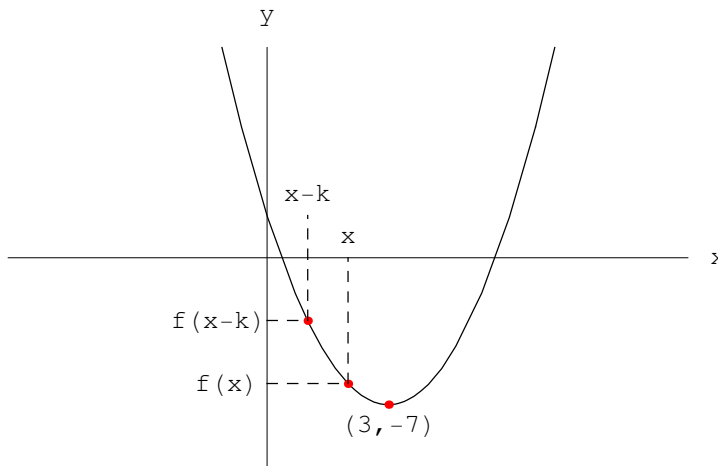
Here's the geometry



The function $f : f(x) = (x - 3)^2 - 7$ is decreasing on $(-\infty, 3)$.

Proof

Suppose $x < 3$ and $k > 0$. Then $x - k < x < 3$. This means $x - k$ is to the left of x and x is to the left of 3. If we can prove that $f(x - k) > f(x)$, then we will have shown that f is decreasing when $x < 3$. In other words, we will have shown that as long as we stay to the left of $x = 3$, the y-coordinate of a point moves down the y-axis as the x-coordinate of the point moves right on the x-axis



We must show that $f(x - k) > f(x)$

$$f(x - k) = (x - k - 3)^2 - 7 \tag{1}$$

$$= [(x - 3) - k]^2 - 7 \tag{2}$$

$$= (x - 3)^2 - 2(x - 3)k + k^2 - 7 \tag{3}$$

$$= -2(x - 3)k + k^2 + (x - 3)^2 - 7 \tag{4}$$

$$= [-2(x - 3)k + k^2] + f(x) \tag{5}$$

$$> f(x) \tag{6}$$

$$\therefore f(x - k) > f(x)$$

□

Here is why line 6 follows from line 5.

Since

$$x < 3,$$

$$x - 3 < 3 - 3 = 0.$$

That is,

$$x - 3 < 0.$$

Thus,

$$(-2)(x - 3)k > 0$$

and

$$[-2(a - 3)k + k^2] > 0.$$

Certainly, a number greater than zero plus $f(x)$ is greater than $f(x)$.

Problems

[1] Prove that $f(x) = x^2 - 6x + 2$ is an increasing function on the interval $(3, \infty)$.